

The magnon kinematics of the AdS/CFT correspondence

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Abstract

The planar dilatation operator of $\mathcal{N} = 4$ supersymmetric Yang-Mills is the hamiltonian of an integrable spin chain whose length is allowed to fluctuate. We will identify the dynamics of length fluctuations of planar $\mathcal{N} = 4$ Yang-Mills with the existence of an abelian Hopf algebra \mathcal{Z} symmetry with non-trivial co-multiplication and antipode. The intertwiner conditions for this Hopf algebra will restrict the allowed magnon irreps to those leading to the magnon dispersion relation. We will discuss magnon kinematics and crossing symmetry on the spectrum of \mathcal{Z} . We also consider general features of the underlying Hopf algebra with \mathcal{Z} as central Hopf subalgebra, and discuss the giant magnon semiclassical regime.

1 Introduction

The appearance of integrable structures on both sides of the AdS/CFT correspondence has played a central role in our current understanding of the duality. The dilatation operator of planar $\mathcal{N} = 4$ supersymmetric Yang-Mills has been shown to correspond, at one-loop for the complete theory [1]-[3], and at several loops in some subsectors [4]-[6], to the hamiltonian of a one-dimensional integrable system. On the string theory side integrability of the classical sigma model on $AdS_5 \times S^5$ [7] allowed a resolution of the theory in terms of spectral curves [8]-[13]. The integral equations satisfied by the spectral density suggested soon after a discrete Bethe ansatz for the quantum string sigma model [14]. The Bethe equations of the gauge theory were then shown to arise from an asymptotic S -matrix in [15], and the S -matrix of $\mathcal{N} = 4$ Yang-Mills was recently derived in [16]. Integrability is thus encoded in a factorizable S -matrix on both sides of the correspondence. The string theory S -matrix describes the scattering of some classical lumps supported on the two-dimensional worldsheet. A semiclassical description at strong 't Hooft coupling of this S -matrix has been proposed in [17] (see also [18]-[24]), based on the classical equivalence of strings moving on $\mathbb{R} \times S^2$ and the sine-Gordon integrable model [25], [26]. On the gauge theory side the S -matrix describing the scattering of magnon fluctuations of the spin chain can be constrained by the symmetries of the system and by the Yang-Baxter triangular equation [16]. However these symmetries are not enough to fix completely the scattering matrix, and additional physical requirements such as unitarity, bootstrap in the case of a non-trivial spectrum of bound states, and crossing symmetry, need to be imposed [27].

The S -matrix for the quantum string Bethe ansatz should extrapolate at weak but finite coupling to the gauge theory S -matrix through a dressing phase factor [14],

$$S_{\text{string}}(p_j, p_k) = e^{i\theta(p_j, p_k)} S_{\text{gauge}}(p_j, p_k) . \quad (1.1)$$

Constraints on this phase factor have been obtained from crossing symmetry in [28]. The dressing factor can also be constrained through comparison with the leading quantum correction to the energy of semiclassical strings [29]-[32]. The quantum dressing factor [31] was in fact shown in [33] to satisfy the crossing equations, and a solution to the crossing relation has recently been proposed in [34].

In many occasions, underlying an integrable system there is a Hopf algebra of symmetries (see for instance [35] and references therein). Factorizable S -matrices can then

always be written in the form

$$S_{12} = S_{12}^0 R_{12} , \quad (1.2)$$

where R_{12} is the intertwiner R -matrix for the Hopf algebra of symmetries, and the factor S_{12}^0 is the dressing phase. The magnons entering into the scattering process correspond to irreps V_{π_i} of the Hopf algebra, and the R -matrix

$$R_{\pi_1 \pi_2} : V_{\pi_1} \otimes V_{\pi_2} \rightarrow V_{\pi_2} \otimes V_{\pi_1} \quad (1.3)$$

is determined by the intertwiner condition

$$R_{\pi_1 \pi_2} \Delta_{\pi_1 \pi_2}(a) = \Delta_{\pi_2 \pi_1}(a) R_{\pi_2 \pi_1} , \quad (1.4)$$

with $\Delta(a)$ the co-multiplication for an arbitrary element a in the Hopf algebra. The physical meaning of the co-multiplication is to provide the composition law that defines the action of symmetry transformations on multimagnon states. From the intertwiner condition (1.4) the non-triviality of the R -matrix follows as a consequence of non-symmetric co-multiplications, *i.e.* “non-classical” composition laws. In the case of the planar limit of $\mathcal{N} = 4$ Yang-Mills, the S -matrix is derived by imposing the intertwiner condition (1.4) with a non-symmetric co-multiplication for the generators of the $SU(2|2)$ algebra [16].

In case the underlying Hopf symmetry algebra contains a non-trivial central Hopf subalgebra [36], new interesting features appear. In particular, not all magnon irreps are allowed, and there is not a universal R -matrix such that

$$R_{\pi_1 \pi_2} = \pi_1 \otimes \pi_2 R . \quad (1.5)$$

In fact, independently of what the intertwiner R -matrix is, we should require, for any element a in the central Hopf subalgebra, that

$$\Delta_{12}(a) = \Delta_{21}(a) . \quad (1.6)$$

This condition, with a non-symmetric co-multiplication, restricts the allowed irreps, that are now parameterized by the eigenvalues of the central elements, to live on certain Fermat curves in the spectrum of the central Hopf subalgebra.¹ In this note we will identify a central Hopf symmetry subalgebra \mathcal{Z} for planar $\mathcal{N} = 4$ Yang-Mills, with three generators and

¹These constraints on the allowed irreps have important implications for the existence of a universal R -matrix. In fact, $R_{\pi_1 \pi_2}$ as defined in (1.5) should in principle exist for any couple of irreps, independently of whether they satisfy condition (1.6).

non-symmetric co-multiplications that will restrict the allowed magnon irreps to those with the BMN-like dispersion relation found in [16] from supersymmetry. Multimagnon physical states are then defined as those invariant under the central Hopf subalgebra leading to the total zero momentum Virasoro condition. The Yang-Mills coupling enters through the undetermined constants parameterizing the Fermat curves that solve condition (1.6). The central Hopf subalgebra governing the dispersion relation for magnons is generated by the two central elements introduced in [16], and an additional central generator K related to the magnon momentum. This Hopf subalgebra is isomorphic to the central Hopf subalgebra of $\mathcal{U}_q(\widehat{SL(2)})$ with q a root of unity. It is known that $\mathcal{U}_q(\widehat{SL(2)})$ is the affine Hopf symmetry algebra of the sine-Gordon model [37], with q determined by the sine-Gordon coupling. At the special value of q a root of unity a central Hopf subalgebra, isomorphic to the one that we have identified for $\mathcal{N} = 4$ Yang-Mills, is dynamically generated. In this case a new dispersion relation for the sine-Gordon solitons can be derived from the central elements in a way analogous to the one we have used in order to reconstruct the magnon dispersion relation.

Finally, let us just briefly comment on the crossing transformation. It was originally suggested in [38] that crossing for an affine Hopf algebra could be defined by promoting the action of the antipode into a certain change in the affine spectral parameter, the “crossing transformation”, that becomes an inner automorphism of the algebra. This, together with the property of the universal R -matrix

$$(\gamma \otimes \mathbb{1})R = R^{-1} , \tag{1.7}$$

leads to a purely algebraic implementation of crossing symmetry. This program was developed for the sine-Gordon model in [37] by imposing invariance under the Drinfeld quantum double $\mathcal{D}(\mathcal{A}, \mathcal{A}^*)$ [39], with \mathcal{A} the quantum affine Hopf algebra of the sine-Gordon model, and \mathcal{A}^* the dual algebra. In [28] this approach was suggested as a way to define crossing for $\mathcal{N} = 4$ Yang-Mills. However, for $\mathcal{N} = 4$, as well as for other integrable models enjoying invariance under a non-trivial central Hopf subalgebra, as for instance the chiral Potts model, an intrinsically algebraic definition of crossing transformations can be given independently of the assumption of existence of a universal R -matrix. The idea is simply to realize that the rapidity plane is the submanifold in the spectrum of the central subalgebra defined by the intertwiner conditions. Therefore, as a simple application of Schur’s lemma we can lift to the rapidity plane the action of the antipode on the generators of the

central subalgebra, and thus define crossing transformations to be this lifted action of the antipode γ on the rapidity plane.

The article is organized as follows. In section 2 we will translate the dynamic $SU(2|3)$ chain into the existence of an abelian central Hopf subalgebra of symmetries, \mathcal{Z} , with non-trivial co-multiplication rules and antipodes. We will show how the generators of this algebra turn out to be the central elements added to $SU(2|2)$ in order to induce the $SU(2|3)$ dynamics. We will then describe how the possible irreps, parameterized by the eigenvalues of the generators of the central algebra, are constrained from intertwiner conditions. These conditions determine the dispersion relation for magnons, and the elliptic curve on the rapidity plane. In section 3 we will wonder about the underlying Hopf algebra with \mathcal{Z} as central subalgebra. We will provide evidence that it should correspond to some quantum Hopf affine algebra at a root of unity, with the central Hopf subalgebra \mathcal{Z} as the enlarged center at such a root of unity. In section 4 we will identify the special features of the magnon kinematics with the conditions for the existence of a non-trivial center for the underlying quantum affine symmetry of the sine-Gordon model.

2 The central Hopf subalgebra

In this section we will describe the geometry underlying the Hopf algebra symmetry of the central extensions of the integrable $SU(2|2)$ chain. In particular, we will find the origin of the Virasoro constraints and the dispersion relation on purely algebraic grounds. But before doing that we will review in some detail the $SU(2|3)$ dynamic chain, and a convenient choice of representations.

2.1 Dynamics and representations

The S -matrix of planar $\mathcal{N} = 4$ supersymmetric Yang-Mills can be constructed using the fact that the complete $PSU(2, 2|4)$ algebra splits into two equal pieces. Both $SU(2|2)$ factors share a central charge that behaves as the hamiltonian, which is part of the symmetry algebra. But in order to deal with the dynamical properties of the chain two extra central elements to the $SU(2|2)$ algebra need to be introduced [16]. These additional central elements act trivially on physical states of vanishing total momentum. However they act in a non-trivial way on the magnon constituents of the physical states and therefore are

relevant to fix the scattering S -matrix.²

Before constructing suitable representations for the $SU(2|2)$ spin chain, let us first recall the $SU(2|3)$ integrable system [5]. In the $SU(2|3)$ sector we have 12 supercharges, Q_α^i and G_α^i , with $i = 1, 2, 3$ and $\alpha = 1, 2$, three complex scalar fields and two spinors. The algebra is enlarged with a $U(1)$ subalgebra generated by the interacting hamiltonian. The full and the interacting hamiltonians, H and δH , satisfy

$$\begin{aligned} [H, Q] &= \frac{1}{2}Q, & [H, G] &= -\frac{1}{2}G, \\ [\delta H, Q] &= 0, & [\delta H, G] &= 0. \end{aligned} \quad (2.1)$$

The dynamical nature of the $SU(2|3)$ chain arises because we can find states with the same quantum numbers and energy, but different length, which is defined in terms of the number of constituents. Fluctuations between these states make the length of the chain a dynamical variable. In particular, the allowed fluctuations, between $\phi^{[1}\phi^2\phi^3]$ and $\psi^{[1}\psi^2]$, play a fundamental role in the definition of the different irreps.

In order to construct the irreps, let us for instance consider two supercharges Q_1^3 and Q_2^3 , acting on the field ϕ^3 . They transform ϕ^3 into ψ^1 and ψ^2 , respectively. We will now act with $Q_1^3 Q_2^3$ on a formal state $|\phi^3\phi^3\rangle$, and transform then the resulting state $|\psi^{[1}\psi^2]\rangle$ into $|\phi^{[1}\phi^2\phi^3]\rangle$ through a fluctuation. After these formal manipulations, we remove one ϕ^3 field from the original and final states, and read from this the action $Q_1^3|\psi^2\rangle$ as $|\phi^{[1}\phi^2]\rangle$.

Now, we will move to the $SU(2|2)$ sector, with only 8 supercharges Q_α^a and G_α^a , where now $a = 1, 2$ and $\alpha = 1, 2$. In this case we only have two scalar fields, and therefore there are no allowed fluctuations. However we can still rely formally on the dynamics of the $SU(2|3)$ chain to construct irreps. Consider for instance the two supercharges $Q_1^1 Q_2^1$ acting on the scalar field ϕ^1 . Through the same argument as above we can consider $Q_1^1 Q_2^1 |\phi^1\phi^1\rangle$, which leads to $|\psi^{[1}\psi^2]\rangle$. Using a fluctuation we transform now this state into $|\phi^{[1}\phi^2 Z]\rangle$, where the field Z is playing the role of ϕ^3 . After this we remove ϕ^1 from the first and last states to obtain $Q_1^1|\psi^2\rangle \simeq |\phi^2 Z\rangle$ or, in general [16],

$$Q_\alpha^i |\psi^\beta\rangle \simeq \epsilon_{\alpha\beta} \epsilon^{ij} |\phi^j Z\rangle. \quad (2.2)$$

Similar formal manipulations, with the spectator field Z , can be employed to define the irreps for the remaining set of supercharges, G_α^a . We can act on ψ^1 with G_1^1 and G_1^2 to

²From a physical point of view the role of this center is very similar to the one played by the center \mathbb{Z}_N of $SU(N)$ in QCD. Physical states are singlets with respect to the center, but the quark constituents transform non-trivially under \mathbb{Z}_N .

produce, respectively, ϕ^1 and ϕ^2 . Thus, $G_1^1 G_1^2 |\psi^1 \psi^1\rangle$ leads to $|\phi^1 \phi^2\rangle$, and we can now define a fluctuation relating $|\phi^1 \phi^2\rangle$ and $|\psi^1 \psi^2 Z^{-1}\rangle$. Using this we get $G_1^1 |\phi^2\rangle \simeq |\psi^2 Z^{-1}\rangle$ or, in general [16],

$$G_\alpha^a |\phi^b\rangle \simeq \epsilon_{\alpha\beta} \epsilon^{ab} |\psi^\beta Z^{-1}\rangle . \quad (2.3)$$

A direct consequence of these dynamic irreps for $SU(2|2)$ is the existence of central terms. In fact, we find that

$$Q_\alpha^a Q_\beta^b |\Psi\rangle \simeq \epsilon_{\alpha\beta} \epsilon^{ab} |\Psi Z\rangle , \quad (2.4)$$

for any generic state $|\Psi\rangle$. This action defines in a natural way a central term B in $SU(2|2)$, because with respect to this algebra $|\Psi\rangle$ and $|\Psi Z\rangle$ are indeed the same state. However, as we will discuss in section 3, the co-multiplication of this central term, and also of the central element R associated to the G_α^a supercharges, is asymmetric and non-trivial. We will use this observation to translate the $SU(2|3)$ dynamics into a deformed co-multiplication for a central Hopf subalgebra.

2.2 Dynamical co-multiplication rules

Let us now introduce the $SU(2|2)$ symmetry algebra. It is generated by two bosonic generators, R_b^a and \mathcal{L}_β^α , together with the supersymmetry generators Q_b^α and G_β^a with central charge c ,

$$\{Q_a^\alpha, G_\beta^b\} = \delta_a^b \mathcal{L}_\beta^\alpha + \delta_\beta^\alpha R_a^b + \delta_a^b \delta_\beta^\alpha c . \quad (2.5)$$

Following [16], we will extend the algebra with two central charges B and R ,

$$\begin{aligned} \{Q_a^\alpha, Q_b^\beta\} &= \epsilon^{\alpha\beta} \epsilon_{ab} B , \\ \{G_\alpha^a, G_\beta^b\} &= \epsilon^{ab} \epsilon_{\alpha\beta} R . \end{aligned} \quad (2.6)$$

The first thing to be noticed concerning these additional central elements is that they define an action on multiple magnon states with a non-trivial and asymmetric co-multiplication. In order to exhibit this co-multiplication, a new element needs to be introduced in the algebra through the relation

$$K|\Psi\rangle = |Z\Psi\rangle , \quad (2.7)$$

where $|\Psi\rangle$ denotes a generic magnon state. Following [16], an excitation with a given momentum p will be

$$|\Psi\rangle = \sum_n e^{ipn} |\dots Z \dots \Psi_n \dots Z \dots\rangle . \quad (2.8)$$

Therefore, inserting or removing a field Z on the excited state will correspond to

$$|Z^\pm \Psi\rangle = \sum_n e^{ipn} |\dots Z \dots \Psi_{n\pm 1} \dots Z \dots\rangle = e^{\mp ip} |\Psi\rangle , \quad (2.9)$$

and thus we find

$$K^{\pm 1} |\Psi\rangle = z^{\pm 1} |\Psi\rangle , \quad (2.10)$$

with z the eigenvalue $z \equiv e^{-ip}$. It is immediate to check now that K commutes with all the generators of $SU(2|2)$, and thus belongs to the center of the algebra. With this new operator we easily find the following co-multiplications for B , R and K ,³

$$\begin{aligned} \Delta B &= B \otimes K + \mathbb{1} \otimes B , \\ \Delta R &= R \otimes \mathbb{1} + K^{-1} \otimes R , \\ \Delta K &= K \otimes K . \end{aligned} \quad (2.11)$$

The operators B , R and K define with the co-multiplication (2.11) an abelian Hopf subalgebra, that we will denote by \mathcal{Z} .

2.3 Spec \mathcal{Z} geometry and the dispersion relation

We will now use invariance under the central Hopf subalgebra \mathcal{Z} as a first step to relate integrability in the planar limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills with the existence of an underlying Hopf algebra symmetry. Let us first show how invariance under the central subalgebra already implies non-trivial constraints on the allowed intertwiners. We will assume, independently of what the underlying Hopf algebra governing the integrable structure of $\mathcal{N} = 4$ Yang-Mills is, that the subalgebra \mathcal{Z} is part of its center. We can now read from the co-multiplications (2.11) that the central Hopf subalgebra must be equipped with an antipode

$$\begin{aligned} \gamma(B) &= -BK^{-1} , \\ \gamma(R) &= -KR , \\ \gamma(K) &= K^{-1} . \end{aligned} \quad (2.12)$$

³In all our equations we implicitly use a graded multiplication defined by

$$(a \otimes b)(c \otimes d) = (ac \otimes bd)(-1)^{[b][c]} .$$

Notice that the antipode is non-trivial because of the non-trivial co-multiplication implied by the dynamics of the spin chain. Using now Schur's lemma we will characterize each irrep by the eigenvalues of the generators of \mathcal{Z} ,

$$\pi(B) = x, \quad \pi(R) = y, \quad \pi(K) = z. \quad (2.13)$$

Next we will introduce a manifold $\text{Spec } \mathcal{Z}$ as the spectrum of \mathcal{Z} [36], and use Schur's lemma to construct a map from the space of irreps into $\text{Spec } \mathcal{Z}$. Now, given two different irreps parameterized by (x_1, y_1, z_1) and (x_2, y_2, z_2) , existence of an intertwiner requires

$$\Delta_{12}(a) = \Delta_{21}(a), \quad \forall a \in \mathcal{Z}. \quad (2.14)$$

When the co-multiplication is non-trivial this condition leads to relations between both irreps. Using now the co-multiplication (2.11) and the map into $\text{Spec } \mathcal{Z}$ defined by (2.13), condition (2.14) leads to the following set of curves of Fermat type in $\text{Spec } \mathcal{Z}$,

$$\frac{x}{z-1} = \alpha, \quad \frac{y}{z^{-1}-1} = \beta, \quad (2.15)$$

with α and β some undetermined constants. Intertwiners will then only exist for irreps satisfying (2.15). In the notation of reference [16], we have $x = ab$ and $y = cd$. Notice that condition (2.15) are precisely those introduced in [16] on the eigenvalues of the central elements B and R , $ab = \alpha(e^{ip} - 1)$ and $cd = \beta(e^{-ip} - 1)$, respectively. These relations arise in [16] by imposing invariance under the central elements of multi-magnon physical states, with vanishing total momentum. However, these relations have a meaning of their own, independently of the condition of vanishing total momentum: they determine the explicit form of the single magnon dispersion relation. Let us also stress that in the previous derivation we have identified the origin of these relations directly from the structure of the central Hopf algebra and the intertwiner condition. The origin of (2.15) is thus independent of the condition of vanishing total momentum on physical states.⁴ In fact, this condition simply means that physical states are singlets with respect to the central algebra \mathcal{Z} , in the same way as in QCD physical states are singlets under \mathbb{Z}_N . The dependence on the Yang-Mills coupling constant appears through the arbitrary constants α and β characterizing the intertwiner Fermat curve. In order to recover the BMN scaling formula [40] the choice $\alpha\beta = 2g^2$ needs to be done [16].

⁴Notice also that the interpretation of the central elements as gauge transformations, $B|\Psi\rangle = \alpha(K|\Psi) - |\Psi\rangle$, $R|\Psi\rangle = \beta(K^{-1}|\Psi) - |\Psi\rangle$ is only valid once we have imposed the Virasoro constraints (2.15).

We can now use the intertwiner condition (2.15) together with the constraint imposed by the closure of $\{Q, G\}$, $ad - bc = 1$, to solve for the central extension,

$$c(z) = \pm \frac{1}{2} \sqrt{1 + 4\alpha\beta(2 - z - z^{-1})} . \quad (2.16)$$

The region in $\text{Spec } \mathcal{Z}$ on which intertwiners for arbitrary pairs of points exist is thus the branch cover of the z -plane defined by the function $c(z)$. In fact irreps for which an intertwiner exist are characterized by the pair $(z, \pm c(z))$. The plus sign will correspond to irreps for particles, and the minus sign to antiparticle irreps.

2.4 The kinetic plane

We will now parameterize different irreps using z and $c(z)$ as coordinates, and we will identify the two possible branches of $c(z)$ with particle and antiparticle irreps. Let us first translate these coordinates into the ones employed in [16], x^\pm . We get

$$x^-(z) = \frac{i}{2g} \frac{1 \pm 2c(z)}{(z - 1)} , \quad (2.17)$$

while $x^+ = zx^-(z)$, which together with $c(z)$ define a double covering of the z -plane. At the self-dual point $z = 1$ with respect to the antipode transformation, $z \rightarrow 1/z$, x^\pm goes to infinity for the positive branch of $c(z)$, or to zero for the negative branch. The magnon charges are now given by

$$q_r(z) = \frac{i(-2ig)^{r-1}}{r-1} \left(\frac{z-1}{1 \pm 2c(z)} \right)^{r-1} \left(\frac{1}{z^{r-1}} - 1 \right) , \quad (2.18)$$

which provides two different values, q_r^\pm , depending on the choice of branch for $c(z)$. They correspond to the magnon charges for particles and antiparticles, respectively. Moving from one particle irrep into an antiparticle irrep amounts to a change in z along a path going through the branch cuts of $c(z)$. In particular, the branch points of $c(z)$ are located at

$$z^\pm = \frac{1}{8\alpha\beta} [1 + 8\alpha\beta \pm \sqrt{1 + 16\alpha\beta}] . \quad (2.19)$$

Together with $z = 0$, the branch points $z = z^\pm$ define an elliptic curve. When written in Weierstrass form,

$$y^2 = 4x^3 - g_2x - g_3 , \quad (2.20)$$

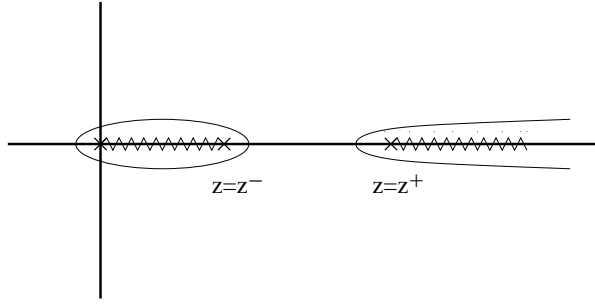


Figure 1: The algebraic curve $y^2 = z(z - z^-)(z - z^+)$.

the elliptic invariants are

$$\begin{aligned} g_2 &= \frac{1}{12}(1 + 16\alpha\beta + 16\alpha^2\beta^2) , \\ g_3 &= \frac{1}{216}(1 + 8\alpha\beta)(-1 - 16\alpha\beta + 8\alpha^2\beta^2) . \end{aligned} \quad (2.21)$$

This is precisely the curve derived in [28] in order to implement crossing on a generalized rapidity plane. In the strong coupling regime $\alpha\beta \rightarrow \infty$ the branch points $z^\pm \rightarrow 1$, and the curve degenerates to

$$y^2 = z(z - 1)^2 . \quad (2.22)$$

Let us now discuss the different magnon irreps in the strong and weak coupling regimes. We will use x , y and z , subject to the constraint (2.15), to parameterize the diverse irreps. And we will employ z as the fundamental parameter, without assuming e^{ip} as a particular representation. We first consider the strong coupling regime defined by $\alpha, \beta \gg 1$. When both α and β are large there are two possible irreps: those with generic x and y eigenvalues, but with z close to 1, and those with generic values of z , while x and y are taken to be large. We will refer to these representations as of type I and type II, respectively. The magnon energy for irreps of type I is

$$c^I(z) = \pm \frac{1}{2} \sqrt{1 + xy} , \quad (2.23)$$

which is finite for both particle and antiparticle irreps. If we now read z as e^{ip} , type I irreps are those with momentum p close to zero. In the case of irreps of type II the momentum p can reach generic values, but the magnon energy is

$$c^{II}(z) \simeq \pm \sqrt{xy} . \quad (2.24)$$

The existence of these two kinds of irreps is crucial in order to understand the strong coupling regime of semiclassical strings. As we reach the strong coupling regime the genuine BMN irreps, with generic and finite values of x and y , and $z \simeq 1$, will start competing with those with generic z , but large values of x and y . However, those irreps with arbitrary values of z will be decoupled in the strong coupling limit, because its energy will diverge, while the ones with $z \simeq 1$ have finite energies. Thus the contribution of irreps of type I is the most natural one to the strong coupling region. Notice also that in this approach the coupling constant is a free parameter labeling different Fermat curves in $\text{Spec } \mathcal{Z}$. Thus, the condition of strong coupling simply fixes a certain curve with large values of α and β . All points on this curve are natural contributions to the strong coupling limit of the S -matrix.

We will now analyze the weak coupling regime, where both α and β are close to zero. In this region, irreps satisfying the intertwiner condition (2.15) are those with generic values of z , but with x and y close to zero. For all of these irreps the energy is fixed at $c(z) = \pm 1/2$. Therefore, the magnon charges (2.18) for the antiparticle irreps will diverge, something that we can interpret in terms of decoupling of antiparticles from the physical spectrum. In the weak coupling region we also find irreps with z close to one, but with x and y both equal or close to zero.

2.5 The $SU(1|2)$ S -matrix and crossing transformations

As it was shown in [16], the $SU(2|2)$ S -matrix can be directly derived from the S -matrix for the $SU(1|2)$ sector. In this section we will briefly discuss the derivation of the $SU(1|2)$ S -matrix, and the meaning of the crossing transformations. Let us consider two irreps V_{π_1} and V_{π_2} of $SU(1|2)$, and let us define the R -matrix as

$$R(1, 2) = \sum_i S^i(1, 2) P_{1,2}^i, \quad (2.25)$$

where $P_{1,2}^i$ are the projector intertwiners, and with the sum extending over the V_{π_i} irreps in the decomposition of $V_{\pi_1} \otimes V_{\pi_2}$. In order to fix the functions $S^i(1, 2)$ we will impose the intertwiner condition

$$S(1, 2) \Delta_{\pi_1 \pi_2}(a) = \Delta_{\pi_2 \pi_1}(a) S(1, 2), \quad (2.26)$$

for any element in the $SU(1|2)$ algebra, and where $S(1, 2) = PR(1, 2)$. The solution to (2.26) if we consider symmetric co-multiplications, $\Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a$, is the trivial one $R(1, 2) = S^0(1, 2) \sum_i P_{1,2}^i = S^0(1, 2) \mathbb{1}$. Since we are interested in the $SU(2|2)$ S -matrix, we

can try to fix the functions $S^i(1, 2)$ in (2.25) by imposing the intertwiner condition (2.26), but for those elements on $SU(2|2)$ that are not in $SU(1|2)$. Denoting by \tilde{Q} and \tilde{G} the additional supersymmetry generators, we can recover the $SU(2|2)$ S -matrix by imposing (2.26) with

$$\begin{aligned}\Delta(\tilde{Q}) &= \tilde{Q} \otimes K + \mathbb{1} \otimes \tilde{Q} , \\ \Delta(\tilde{G}) &= \tilde{G} \otimes K^{-1} + \mathbb{1} \otimes \tilde{G} ,\end{aligned}\tag{2.27}$$

where the generator K is the one introduced in (2.7).

It is worth to compare this construction with the one in [41]. In this reference the affinization of the R -matrix for a quantum deformation of the $SU(1|2)$ group was considered. The R -matrix was constrained by two conditions, which are the intertwiner condition (2.26) for the generators of $SU(1|2)$ with a deformed co-multiplication, and an additional intertwiner condition involving the extra generators of the affine algebra. This new condition depends on the spectral parameter, which turns to be the rapidity. The main difference with the previous construction is that the additional intertwiner conditions in the case of $SU(2|2)$ are not associated with any form of affinization. Moreover, the magnon rapidities enter into the R -matrix through the co-multiplication (2.27) leading to an S -matrix depending on the two magnon rapidities.

Let us now consider the issue of crossing. As we have already discussed in the previous subsections, due to the existence of a central Hopf subalgebra \mathcal{Z} the different magnon irreps can be characterized by the eigenvalues of the central elements. In this case, we can lift the action of the antipode on the generators of \mathcal{Z} to the space of irreps. We thus define

$$\bar{\pi}(a) = \pi(\gamma(a)) , \quad \forall a \in \mathcal{Z} .\tag{2.28}$$

This map leads to

$$\begin{aligned}x &\rightarrow \bar{x} = -z^{-1}x , \\ y &\rightarrow \bar{y} = -zy , \\ z &\rightarrow \bar{z} = z^{-1} .\end{aligned}\tag{2.29}$$

Notice now that the identification of $\text{Spec } \mathcal{Z}$ with the rapidity plane allows a kinematical definition of crossing as transformation on the rapidity plane which is completely determined by the definition of the antipode for the generators of \mathcal{Z} . In this sense, this

implementation of crossing only depend on the central Hopf subalgebra. This definition of crossing on the space of irreps is completely independent of the existence of a universal R -matrix. We can represent the previous implementation of crossing through

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\pi} & \text{Spec } \mathcal{Z} \\ \gamma \downarrow & & \downarrow \text{cross} \\ \mathcal{Z} & \xrightarrow{\pi} & \text{Spec } \mathcal{Z} \end{array} \quad (2.30)$$

where π is the map from the central subalgebra into the rapidity manifold given by Schur's lemma, γ is the antipode in \mathcal{Z} and “cross” is the crossing transformation. Notice that cross preserves the curves obtained from the intertwiner condition.

3 The Hopf algebra symmetry

In the previous section we have identified an abelian Hopf subalgebra \mathcal{Z} that encodes the information on the multi-magnon states of vanishing total momentum through the intertwiner conditions. In order to construct the Hopf subalgebra \mathcal{Z} we have employed the central elements of $SU(2|2) \ltimes \mathbb{R}^2$, together with the generator K . In this section we will pose the question of the existence of a Hopf algebra \mathcal{A} with central extension \mathcal{Z} , in such a way that points in $\text{Spec } \mathcal{Z}$ are in a one-to-one correspondence with irreps of \mathcal{A} .

The most natural candidate to \mathcal{A} , which cannot a priori be identified with $SU(2|2) \ltimes \mathbb{R}^2$, is a quantum group Hopf algebra, with the quantum deformation parameter at a particular root of unity, because quantum groups at roots of unity exhibit an enlarged central Hopf subalgebra. Let us consider to clarify ideas a Hopf algebra with generators E_i, F_i and K_i in the Cartan-Chevalley basis, and with $q^l = 1$. In this case E_i^l, F_i^l and K_i^l are part of the generators of the central Hopf subalgebra. It is important to stress that these central elements are not added to the algebra \mathcal{A} , but rather arise as a consequence of the quantum deformation parameter q being a root of unity. With this observation in mind, what we are searching for must be a Hopf algebra and a particular value of $q^l = 1$, such that the corresponding central subalgebra is isomorphic to the one defined through (2.11) in the $\mathcal{N} = 4$ Yang-Mills case. This would provide a natural explanation on the origin of the additional central elements extending the $SU(2|2)$ algebra.

3.1 Cyclic two-dimensional irreps

In order to uncover the algebra \mathcal{A} , let us first consider the kind of periodic irreps that we can define in $SU(2|2)$. We can construct two-dimensional subspaces by the following cycle of transformations,

$$\dots \xrightarrow{Q_\beta^j} |\phi^i\rangle \xrightarrow{Q_\alpha^i} |\psi^\alpha\rangle \xrightarrow{Q_\beta^j} |\phi^i\rangle \xrightarrow{Q_\alpha^i} \dots , \quad (3.1)$$

with $i \neq j$ and $\alpha \neq \beta$. In a similar way we can define

$$\dots \xrightarrow{G_\beta^j} |\psi^\alpha\rangle \xrightarrow{G_\alpha^i} |\phi^i\rangle \xrightarrow{G_\beta^j} |\psi^\alpha\rangle \xrightarrow{G_\alpha^i} \dots \quad (3.2)$$

It is of course very tempting, once we have these two-dimensional vector spaces, to interpret them as some sort of cyclic irreps. In particular, we will identify them with the two-dimensional cyclic irreps of $\mathcal{U}_q(\widehat{SL(2)})$ with $q^4 = 1$. In fact, we can define operators

$$\begin{aligned} \tilde{Q}_{\alpha\beta}^{ij} &\equiv Q_\alpha^i + Q_\beta^j , \\ \tilde{G}_{\alpha\beta}^{ij} &\equiv G_\alpha^i + G_\beta^j , \end{aligned} \quad (3.3)$$

satisfying

$$\begin{aligned} (\tilde{Q}_{\alpha\beta}^{ij})^2 &= \epsilon^{ij} \epsilon_{\alpha\beta} B , \\ (\tilde{G}_{\alpha\beta}^{ij})^2 &= \epsilon^{ij} \epsilon_{\alpha\beta} R . \end{aligned} \quad (3.4)$$

Then, the cyclic representation can be chosen as

$$\begin{aligned} \dots \xrightarrow{\tilde{Q}_{21}^{12}} |\phi^1\rangle \xrightarrow{\tilde{Q}_{21}^{12}} |\psi^2\rangle \xrightarrow{\tilde{Q}_{21}^{12}} |\phi^1\rangle \xrightarrow{\tilde{Q}_{21}^{12}} \dots , \\ \dots \xrightarrow{\tilde{G}_{21}^{12}} |\psi^2\rangle \xrightarrow{\tilde{G}_{21}^{12}} |\phi^1\rangle \xrightarrow{\tilde{G}_{21}^{12}} |\psi^2\rangle \xrightarrow{\tilde{G}_{21}^{12}} \dots . \end{aligned} \quad (3.5)$$

In this way we can think of (3.5) as a cyclic irrep of $\mathcal{U}_q(\widehat{SL(2)})$ with $q^4 = 1$, where $\tilde{Q}_{21}^{12} \sim E$ and $\tilde{G}_{21}^{12} \sim F$, and with B and R being parts of the central Hopf subalgebra of $\mathcal{U}_q(\widehat{SL(2)})$ at $q^4 = 1$. This should be just considered as a formal hint toward the challenge of uncovering the underlying Hopf symmetry algebra \mathcal{A} whose central subalgebra \mathcal{Z} is the central Hopf subalgebra of $\mathcal{N} = 4$ Yang-Mills. From (3.4) it follows that $\tilde{Q}_{\alpha\beta}^{ij} = \tilde{Q}_{\beta\alpha}^{ji}$, so that there are only two different \tilde{Q} operators, \tilde{Q}_{21}^{12} and \tilde{Q}_{12}^{21} . The same holds true for the \tilde{G} operators. Thus, we find the adequate number of operators for the map to the affine $\mathcal{U}_q(\widehat{SL(2)})$ at $q^4 = 1$, which therefore appears as a natural candidate to at least part of the underlying Hopf algebra of the planar limit of $\mathcal{N} = 4$ Yang-Mills.

As a step further, we will also suggest a co-multiplication for the supersymmetry generators Q and G , consistent with the one defining \mathcal{Z} . Dynamics was introduced in representation theory through fluctuations of the form (2.4). A suitable formal way to respect these fluctuation equivalences would be the definition of irreps of the type $Q_\beta^i \phi^i \sim \psi^\beta Z^{1/2}$, and $Q_\alpha^i \psi^\beta \sim \phi^i Z^{1/2}$. The advantage of these definitions of irreps is formal consistency with the fact that the central elements B and R have non-trivial co-multiplications and, as consequence, non-trivial antipodes. Thus we will formally extend the Hopf algebra structure by requiring

$$\begin{aligned}\Delta Q &= Q \otimes \mathbb{1} + \mathcal{K} \otimes Q , \\ \Delta G &= G \otimes \mathbb{1} + \mathcal{K}^{-1} \otimes G ,\end{aligned}\tag{3.6}$$

where \mathcal{K} would be part of the Cartan subalgebra of \mathcal{A} , and such that $\mathcal{K}^2 = K$, with K the generator in \mathcal{Z} . The corresponding antipodes are

$$\begin{aligned}\gamma(Q) &= -\mathcal{K}^{-1} Q , \\ \gamma(G) &= -\mathcal{K} G .\end{aligned}\tag{3.7}$$

Notice that now (3.6) and (3.7) are perfectly consistent with the co-multiplication and the antipode of \mathcal{Z} , as well as with relations (3.4).

4 Magnon kinematics and the sine-Gordon model

In this section we will explore the kinematics of giant magnons on semiclassical strings. Semiclassical strings moving in $\mathbb{R} \times S^2$ are equivalent to the sine-Gordon integrable model [25, 26]. The Virasoro constraints lead to

$$[\partial_\tau^2 - \partial_\sigma^2] \phi = -\frac{1}{2} \sin(2\phi) .\tag{4.1}$$

This sine-Gordon model corresponds to a particular value of the coupling constant, $\beta = 2$.⁵

⁵We are normalizing the sine-Gordon model as

$$S = \frac{1}{4\pi} \int d^2 z \partial_z \phi \partial_{\bar{z}} \phi + \frac{\lambda}{\pi} \int d^2 z \cos(\beta \phi) .$$

Thus, in the string case we have $\beta = 2$ and $\lambda = \frac{1}{8}$ (*i.e.* $m^2 = 1$).

This giant magnons of [17] correspond precisely to sine-Gordon solitons, with the only difference that the magnon energy goes like the inverse of the soliton energy. Introducing the sine-Gordon rapidity θ , the soliton energy is given by

$$E_{\text{sG}} = \cosh \theta . \quad (4.2)$$

Through the map of the string sigma model into the sine-Gordon system we have

$$\sin \left(\frac{p}{2} \right) = \frac{1}{\cosh \theta} . \quad (4.3)$$

Thus, the magnon energy, given by the large coupling limit of the dispersion relation, $E \simeq \sqrt{\lambda}/\pi \sin(p/2)$, goes like $1/E_{\text{sG}}$. Notice also that the magnon energy $E \simeq 1/\cosh \theta$ is strictly the same as the one of elementary excitations of the anti-ferromagnetic isotropic Heisenberg chain in the thermodynamic limit. In fact in this case we have $E(\theta) \simeq 1/\cosh \theta$ and

$$p(\theta) \simeq \pi - \tan^{-1}(\sinh \theta) , \quad (4.4)$$

which leads to the dispersion relation $E(p) \simeq \sin(p/2)$. Let us recall that the low lying excitations for the anti-ferromagnetic chain are the holes on the Dirac sea of Bethe strings, and they have spin $\frac{1}{2}$ [42]. The formal relation with the sine-Gordon model is hidden in the special form of the rapidity dependence of the momentum excitations.

If we move now into weaker coupling and interpret $E(p) = \sqrt{1 + \lambda/\pi^2 \sin^2(p/2)}$ as the relativistic relation $E^2 = m^2 + p^2$, we should identify $\lambda/\pi^2 \sin^2(p/2)$ with the momentum square. Thus, a natural definition of rapidity is [17]

$$\sinh^2 \theta_p = \frac{\lambda}{\pi^2} \sin^2 \left(\frac{p}{2} \right) . \quad (4.5)$$

From this identification we get

$$e^{\theta_p} = \mp 2\sqrt{xy} \pm \sqrt{4xy + 1} \quad (4.6)$$

Notice that in the strong coupling limit and for irreps of type I we have a generic value of the rapidity θ_p .

Let us now try to understand the physical meaning of this rapidity in terms of the quantum symmetries of the sine-Gordon model. It is a well known result that the non-local charges in the sine-Gordon model generate an affine quantum algebra $\mathcal{U}_q(\widehat{SL(2)})$, with

$$q = e^{-\frac{2\pi i}{\beta^2}} , \quad (4.7)$$

so that $q^4 = 1$ when $\beta = 2$. This affine quantum algebra at $q^4 = 1$ is isomorphic to the $\mathcal{N} = 2$ supersymmetry algebra, with generators Q_\pm and \bar{Q}_\pm . We can now represent the generators of $\mathcal{U}_q(\widehat{SL(2)})$ in terms of the ones of $\mathcal{U}_q(SL(2))$, E , F and \mathcal{K} , if we introduce an affine parameter, that plays the role of a rapidity,

$$\begin{aligned} Q_+ &= e^\theta E, & Q_- &= e^\theta F, \\ \bar{Q}_+ &= e^{-\theta} F \mathcal{K}, & \bar{Q}_- &= \mathcal{K}^{-1} e^{-\theta} E. \end{aligned} \quad (4.8)$$

For regular solitonic irreps we get from here the standard relation $p = \sinh \theta$, with p defined in terms of Q_\pm and \bar{Q}_\pm by the Serre relations. These are the standard sine-Gordon solitons that are directly connected at strong coupling with the giant magnons, up to the change in the energy relations. However for $q^4 = 1$ we also have classical irreps, with Q_\pm and \bar{Q}_\pm being non-vanishing elements in the center. For these non-classical irreps we also have $K = \mathcal{K}^2$ in the center of $\mathcal{U}_q(\widehat{SL(2)})$, together with E^2 and F^2 . In this case we can use the elements in the center to construct a candidate for the momentum through [43]

$$P = Q_\pm^2, \quad \bar{P} = \bar{Q}_\pm^2 \quad (4.9)$$

with the physical momentum $p = P - \bar{P}$. If we write $p^2 = (Q_+^2 - \bar{Q}_+^2)(Q_-^2 - \bar{Q}_-^2)$ we get, when $\theta = 0$,

$$p^2 = \frac{\lambda}{\pi^2} (z - 1)(z^{-1} - 1), \quad (4.10)$$

for the eigenvalues of E^2 , F^2 and $K = \mathcal{K}^2$ given by $\sqrt{\lambda}(z - 1)/2\pi$, $\sqrt{\lambda}(z^{-1} - 1)/2\pi$ and z , respectively. This is just the relation $p^2 = \lambda/\pi^2 \sin^2(p/2)$ for $z = e^{-ip}$. This provides further evidence that magnons must be related to sine-Gordon solitons in non-classical irreps where the central subalgebra is realized in a non-trivial way. The kinematic arena for these magnons is determined by the Spec of the central subalgebra of symmetries of $\mathcal{U}_q(\widehat{SL(2)})$ ⁶. Nicely enough, this central subalgebra on the string theory side is isomorphic to the central subalgebra \mathcal{Z} for $\mathcal{N} = 4$ Yang-Mills. Thus, the constituents on both sides of the correspondence share the same kinematic arena, defined by the same central subalgebra. The study of these common features clearly deserves further analysis.

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⁶A similar phenomena takes place in the chiral Potts model [44].

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